Discrete Geometric Mechanics for Variational Time Integrators

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Abstract

In this chapter, we present a *geometric*—instead of a traditional numerical-analytic—approach to the problem of time integration. Geometry at its most abstract is the study of symmetries and their associated invariants. Variational approaches based on such notions are commonly used in geometric modeling and discrete differential geometry. Here we will treat mechanics in a similar way. Indeed, the very essence of a mechanical system is characterized by its *symmetries* and *invariants*. Thus preserving these symmetries and invariants (*e.g.*, certain momenta) into the discrete computational setting is of paramount importance if one wants discrete time integration to properly capture the underlying continuous motion. Motivated by the well-known variational and geometric nature of most dynamical systems, we review the use of *discrete variational principles* as a way to derive robust, and accurate time integrators.

1 Introduction

Prediction is difficult, especially of the future.

—Mark Twain

Time Evolution of Dynamical Systems Time evolving phenomena such as the swinging of a clock pendulum, the bouncing of a soft ball on the floor, or even biological systems and stock market indicators are often modeled (*i.e.*, studied and understood) as dynamical systems. Mathematical models of the evolution in time of these systems generally involve systems of differential equations. *Solving* a physical system means figuring out how to move the system forward in time from a set of initial conditions, allowing the computation of, for instance, the trajectory of the soft ball (*i.e.*, its position as a function of time) thrown onto the floor. Although this example can easily be solved analytically, direct solutions of the differential equations governing a system are generally hard or impossible—we need to resort to *numerical techniques* to find a discrete temporal description of a motion. Consequently, there has been a significant amount of research in applied mathematics on how to deal with some of the most useful systems of equations, leading to a plethora of numerical schemes with various properties, orders of accuracy, and levels of complexity of implementation (see [Press et al. 1992] for a general overview).

Accurate vs. Qualitative Integrators While it is unavoidable to make approximations in numerical algorithms (*i.e.*, to differ from the continuous equivalent), the matter becomes whether the numerics can provide satisfactory results. The notion of satisfactory is, however, objective-dependent. If simulation is used for the design of a plane wing through a series of tests over a wide range of situations, *qualitative* reproduction of the wing behavior may be preferable over absolute numerical *accuracy*. If, however, simulation is used to find the proper launch parameters for a satellite to be put at a particular orbit, accurate results are crucial. This apparent mismatch in objectives has been, historically, aggravated by the cultural gap existing between applied and theoretical communities. We will show that in fact, one does not have to ask for *either* predictability *or* accuracy: simple methods exist that guarantee *good statistical predictability* by respecting the geometric properties of the exact flow of the differential equations, while being *also* easily rendered arbitrarily accurate.

Animation, or Simulation? In Computer Animation, time integrators are crucial computational tools at the core of most physics-based animation techniques. Animating a rigid body for instance uses the principles of classical mechanics, involving second order differential equations. In their most rudimentary form, these principles express the *relationship between forces acting on the body and its acceleration* given by *Newton's laws of motion*. From these equations of motion, classical time integrators (such as fourth-order Runge-Kutta, implicit Euler, and more recently the Newmark scheme) have been methods of choice in practice [Parent 2001; Hauth et al. 2003] to result in motions with good visual behavior—arguably, the top priority in graphics. Nonetheless, allowing the equations of motion to be slightly violated is commonly used to better control the resulting animation [Barzel et al. 1996], as long as it still looks visually plausible. In other words, local accuracy can be tinkered with just as long as the motion is still "globally" right.

Goals In this chapter, we provide an introduction to geometric mechanics, first from a continuous, then from a discrete point of view. Departing sharply from traditional numerical-analytic expositions, we point out how respecting the geometry of mechanics is not only natural, but it provides simple and powerful foundations for the design of robust time integrators. In particular, we will introduce the notion of *variational integrators* as a class of solvers specifically designed to preserve this underlying physical structure, even for large time steps that would produce overdamped or diverging results with more traditional methods.

2 Geometric Approach to Mechanics

Dynamics as a Variational Problem Considering mechanics from a variational point of view goes back to Euler, Lagrange and Hamilton. The form of the variational principle most important for continuous mechanics is due to Hamilton, and is often called *Hamilton's principle* or the *least action principle*: it states that a dynamical system always finds an optimal course from one position to another—or, as P.L. Moreau de Maupertuis put it, "Nature is thrifty in all its actions". A more formal definition will be presented in Section 4.1, but one consequence is that we can recast the traditional way of thinking about an object accelerating in response to applied forces into a geometric viewpoint. There, the path followed by the object has *optimal geometric properties*—analog to the notion of geodesics on curved surfaces. This point of view is equivalent to Newton's laws in the context of classical mechanics, but is broad enough to encompass areas ranging to E&M and quantum mechanics.

Discrete Structure-Preserving Integrators Geometric integrators are a class of numerical time-stepping methods that exploit this geometric structure of mechanical systems [Hairer et al. 2002]. Of particular interest within this class, *variational integrators* [Marsden and West 2001] discretize the variational formulation of mechanics we mentioned above, providing a solution for most ordinary and partial differential equations that arise in mechanics. While the idea of discretizing variational formulations of mechanics is standard for elliptic problems using Galerkin Finite Element methods for instance, only recently has it been used to derive variational time-stepping algorithms for mechanical systems. This approach allows the construction of integrators with any order of accuracy [West 2003; Lew 2003], and can handle constraints as well as external forcing. Results have been shown to be equal or superior to all other types of integrators for simulations of a large range of physical phenomena [Kane et al. 2000], making this discrete-geometric framework both versatile and powerful.

Of particular interest in computer animation, the simplest variational integrator can be implemented by taking two consecutive positions $q_0 = q(t_0)$ and $q_1 = q(t_0+dt)$ of the system to compute the next position $q_2 = q(t_0 + 2dt)$. Repeating this process calculates an entire discrete (in time) trajectory. In this chapter, we describe the foundations necessary to derive such variational schemes based on geometric arguments.

3 A Motivating Example: The Pendulum

Before we delve into the details of what variational integrators are, let us first look at a simple example to exemplify how slight variations in the design of time integrators can result in widely different behaviors.

3.1 Setup and Conventions

Consider a simple pendulum of mass m and length L, swinging under the influence of the gravitational acceleration g. Let $q(t)$ represents the pendulum's angle with the vertical at time t. As this angle is the only degree of freedom for this simple example, we can express the equations of motion for this system based solely on q and its derivatives:

$$\ddot{q} = -\frac{g}{L}\sin q, \tag{1}$$

where we use the "dot" notation to represent derivatives with respect to time, *i.e.*:

$$\dot{q} := \frac{dq}{dt}, \qquad \text{and} \qquad \ddot{q} := \frac{d^2q}{dt^2}.$$

We can rewrite this equation as a system of two *coupled* first-order equations in the variables q and v:

$$\dot{q} = v \tag{2}$$

$$\dot{v} = -\frac{g}{L}\sin q \tag{3}$$

If the initial conditions $q(0)$ and $\dot{q}(0)$ are given, then we could theoretically solve this differential equation for q. Assume for a moment that we don't have access to the analytical solution to this problem (in fact, as in many cases, no such solution is known). We can only hope to *approximate* the solution using an integrator. To achieve this goal, we first discretize the problem. That is, we break up time into N equal steps of length h, so that we no longer have a continuous notion of time, but have instead a discrete set of times $t_k = kh$. Then, finding an approximation to the differential equation on our new discrete time domain is tantamount to solving for the values of the angles at the various times, *i.e.*, finding the values $q_k = q(t_k)$ for $k = 1, \ldots, N$.

Given this setup, how can we compute the q_k's? There are actually many choices, and the important point to realize is, not all of them perform equally well.

3.2 Three Numerical Schemes

Assuming that the time step h is small enough with respect to all other derivatives of q, we could leverage the well-known Taylor expansion:

$$q(t+h) = q(t) + h\dot{q}(t) + \mathcal{O}(h)\,.$$

Using this first order approximation, one can easily derive the following, straightforward update rules by applying Taylor expansion to both q and v:

$$\begin{cases} q_{k+1} = q_k + h\, v_k \\ v_{k+1} = v_k - h\, \dfrac{g}{L}\sin q_k \end{cases}$$

Given the previous values q_k, v_k, this method gives us an explicit formula to compute the next values in time q_{k+1}, v_{k+1}; this specific time integrator is called the ***explicit Euler method***. Repeating this procedure by setting $k := k + 1$ provides a way to compute the whole motion.

Alternatively, we could change the time integration procedure by evaluating the right hand sides of the former rules at the *next* time step, through:

$$\begin{cases} q_{k+1} = q_k + h\, v_{k+1} \\ v_{k+1} = v_k - h\, \dfrac{g}{L}\sin q_{k+1} \end{cases}$$

This method is no longer explicit, but *implicit*: one needs to use a (non-linear) solver to find the pair q_{k+1}, v_{k+1} that satisfy these equations, given the current values q_k and v_k. This time integrator is traditionally called the ***implicit Euler method***.

Finally, one could use a seemingly strange mix of the two, by first updating v_{k+1} explicitly, then q_{k+1} using the new value v_{k+1} (thus, still explicitly):

$$\begin{cases} v_{k+1} = v_k - h\, \dfrac{g}{L}\sin q_k \\ q_{k+1} = q_k + h\, v_{k+1} \end{cases}$$

Notice that the difference with the first scheme is rather minimal. However, this particular time integrator is known as the ***symplectic Euler method***.

These three methods are called *finite difference methods*, since they approximate the left-hand side derivatives of Eqs. (2-3) by taking the difference between consecutive values. Notice in particular that, while the implicit method is more computationally expensive, the two others involve the exact same amount of operations. Thus, their behavior should not be very different, right?

3.3 Comparing Integrators

Numerical tests of these three integrators reveal obvious differences in practice (to avoid going too much into sordid details of numerical analysis, we will stick to a fixed time step $h = 0.01$ for all experiments). First, one quickly realizes that the explicit Euler suffers from stability problems: the motion of the pendulum *amplifies* over time! An obvious consequence is that the pendulum's energy

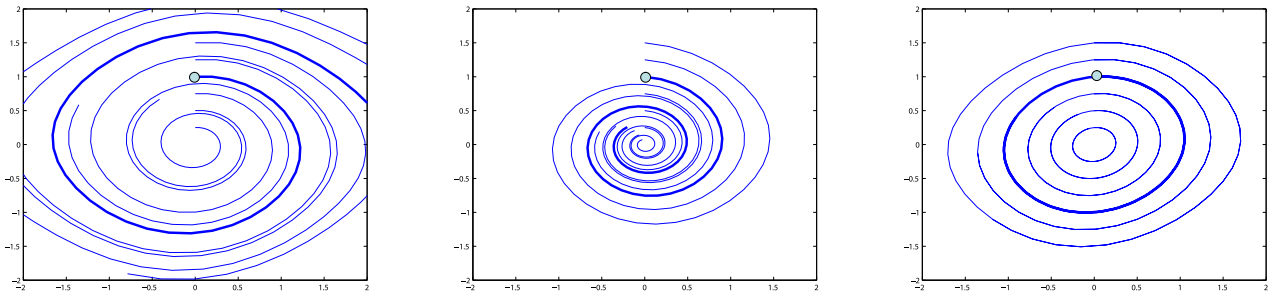

Figure 1: *Three integrators in phase space (q, p): (left) explicit, (middle) implicit, (right) symplectic. Six initial conditions are shown, with their respective trajectories; only the symplectic integrator captures the periodic nature of the pendulum. The bold trajectories correspond to the exact same initial condition.*

increases over time, rather than being conserved. Thus, in practice, the solution often "blows up" and becomes unstable as time progresses—not a great quality for a time integrator. Fortunately, the implicit Euler *is* stable: the amplitude of the pendulum's oscillations actually *decreases* over time, avoiding any chance of numerical divergence (see Fig. 2). However, this stability comes at a cost: the pendulum *loses* energy, causing the pendulum to slow down towards a stop, even if our original equations do not include any damping forces. Effectively, we resolved the stability issue through the introduction of ***numerical dissipation***—but we induced the opposite problem instead. The symplectic method, on the other hand, both is stable *and* oscillates with constant amplitudes. This is obviously a superior method for physical simulation, given that no additional numerical operations were needed to get the correct qualitative behavior!

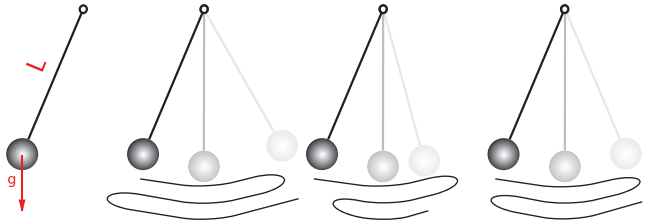

Figure 2: *The pendulum: for the equation of motion of a pendulum of length L and unit mass in a gravitation field g (left), our three integrators behave very differently: while the explicit Euler integrator exhibits amplifying oscillations, the implicit one dampens the motion, while the symplectic integrator perfectly captures the periodic nature of the pendulum.*

Now, if we are only solving for the position of the pendulum only at one particular time, it does not really matter which method we use: taking small enough time steps will guarantee arbitrarily good accuracy. However, if we wish our time integrator to be globally predictive, the least we can ask for is to get a pendulum that actually keeps on swinging. Even a simple animation of a grandfather clock or a child on a swing would look unrealistic if it seemed to gain or lose amplitude inexplicably. In other words, the behavior of energy over time is of key importance. But how do we know that an integrator will have these good properties ahead of time? Can we construct them for an arbitrary physical system? The answer, as we shall see, comes from the world of geometric mechanics and a concept called *symplecticity*.

4 Geometric Mechanics

In the familiar Newtonian view of mechanics, we begin by adding up the forces F on a body and writing the equations of motion using the famous second law,

$$F = ma, \tag{4}$$

where a represents the acceleration of the body. With geometric mechanics, however, we consider mechanics from a variational point of view. In this section, we review the basic foundations of *Lagrangian mechanics*, one of the two main flavors of geometric mechanics (we will only point to some connections with *Hamiltonian mechanics*).

4.1 Lagrangian Mechanics

Consider a finite-dimensional dynamical system parameterized by the *state variable* q, i.e., the vector containing all degrees of freedom of the system. In mechanics, a function of a position q and a velocity $\dot{q}$ called the Lagrangian function L is defined as the kinetic energy K (usually, only function of the velocity) minus the potential energy U of the system (usually, only function of the state variable):

$$L(q, \dot{q}) = K(\dot{q}) - U(q).$$

Variational Principle The *action functional* is then introduced as the integral of L along a path $q(t)$ for time $t \in [0, T]$:

$$S(q) = \int_0^T L(q, \dot{q})\, dt.$$

With this definition, the main result of Lagrangian dynamics, *Hamilton's principle*, can be expressed quite simply: this variational principle states that *the correct path of motion of a dynamical system is such that its action has a stationary value*, *i.e.*, the integral along the correct path has the same value to within first-order infinitesimal perturbations. As an "integral principle" this description encompasses the entire motion of a system between two fixed times (0 and T in our setup). In more ways than one, this principle is very similar to a statement on the *geometry* of the path $q(t)$: the action can be seen as the analog of a measure of "curvature", and the path is such that this curvature is extremized (*i.e.*, minimized or maximized).

Euler-Lagrange Equations How do we determine which path optimizes the action, then? The method is similar to optimizing an ordinary function. For example, given a function $f(x)$, we know that its critical points exist where the derivative $\nabla f(x) = 0$. Since q is a path, we cannot simply take a "derivative" with respect to q; instead, we take something called a ***variation***. A variation of the path q is written δq, and can be thought of as an infinitesimal

perturbation to the path at each point, with the important property that the perturbation is null at the endpoints of the path. Computing variations of the action induced by variations δq of the path $q(t)$ results in:

$$\delta S(q) = \delta \int_0^T L(q(t), \dot{q}(t))\, dt = \int_0^T \left[\frac{\partial L}{\partial q} \cdot \delta q + \frac{\partial L}{\partial \dot{q}} \cdot \delta \dot{q} \right] dt$$
$$= \int_0^T \left[\frac{\partial L}{\partial q} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) \right] \delta q\, dt \; + \; \left[\frac{\partial L}{\partial \dot{q}} \cdot \delta q \right]_0^T ,$$

where integration by parts is used in the last equality. When the endpoints of $q(t)$ are held fixed with respect to all variations $\delta q(t)$ (*i.e.*, $\delta q(0) = \delta q(T) = 0$), the rightmost term in the above equation vanishes. Therefore, the condition of stationary action for arbitrary variations δq with fixed endpoints stated in Hamilton's principle directly indicates that the remaining integrand in the previous equation must be zero for all time t, yielding what is known as the *Euler-Lagrange equations*:

$$\frac{\partial L}{\partial q} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) \; = \; 0. \tag{5}$$

For a given Lagrangian, this formula will give the *equations of motion* of the system.

Forced Systems To account for non-conservative forces or dissipation F, the least action principle is modified as follows:

$$\delta \int_0^T L(q(t), \dot{q}(t))\, dt + \int_0^T F(q(t), \dot{q}(t)) \cdot \delta q\, dt = 0\, .$$

This is known as the *Lagrange-d'Alembert principle*.

Lagrangian vs. Hamiltonian Mechanics. Hamiltonian mechanics provides an alternative formulation, which is closely related to the Lagrangian. The reader may consult any book on mechanics for the relationships between the two descriptions. We simply point out here (as it will be useful later) that in the Hamiltonian formulation, the dynamics are described in *phase space*, i.e, the current state of a dynamical system is given as a pair (q, p), where q is the state variable, while p is the momentum, defined by $p = \partial L / \partial \dot{q}$.

4.2 Example

Let make the previous definitions more concrete by detailing a particularly simple example. Given a particle with mass M in a gravitational field, *i.e.*, in a potential field $V = Mg \cdot q$, the Lagrangian is written:

$$L(q, \dot{q}) = \frac{1}{2} \dot{q}^T M \dot{q} - Mg \cdot q.$$

Taking the variation of the action, one gets:

$$\delta \int_a^b \left(\frac{1}{2} \dot{q}^T M \dot{q} - Mg\, q \right) dt = \int_a^b \left(M\dot{q} \cdot \delta \dot{q} - Mg \cdot \delta q \right) dt.$$

Next, we integrate the $\delta \dot{q}$ term by parts; the boundary terms disappear, since $\delta q = 0$ at the endpoints.

$$= \int_a^b (-M\ddot{q} - Mg) \cdot \delta q\, dt = 0.$$

Since the integral equals 0 for any variation δq, the first term inside the integral must equal 0. Therefore, the Euler-Lagrange equations become:

$$M\ddot{q} = -Mg,$$

which are precisely the Newtonian equation of motion $F = ma$.

4.3 Symmetries and Invariants

Finally, we arrive at a crucial question: why exactly do physical systems conserve certain quantities? If we can answer this question *and* mimic the continuous dynamics in our discrete implementations, only then can we hope to get good numerical properties for our time integrators. This question is partially answered by ***Noether's theorem***, an extremely powerful theorem in physics which states that each symmetry of a system leads to a physical invariant (*i.e.*, a conserved quantity). For example, take the dynamics of an elastic object in the void. The Lagrangian can easily be shown to be translation invariant: translating all the mass particles of the elastic object would not change the value of the Lagrangian. Similarly, the Lagrangian is rotation-invariant as moving all the particles of the object by a global rotation has no reason to affect the Lagrangian either. This means that the system has a *translational and rotational symmetry*. Noether's theorem then states that the *linear and angular momenta* are preserved. These symmetries, if respected in the discrete setting, will provide equivalent discrete invariants in time integrators! In fact, we will see that these invariants can be preserved in time integrators at no extra computational cost by simply respecting the geometric, variational nature of dynamics.

4.4 Phase Space and Symplecticity

To visualize a dynamical system, we often plot its trajectories in ***phase space***. In its simplest version as in the one-dimensional pendulum case, it is in fact a phase plane where one axis represents the position q and the other axis represents either velocity $\dot{q}$ or, more usually, momentum $p = m\dot{q}$. Note that for higher dimensional systems, there is an additional axis corresponding to each additional position component q^i and its corresponding velocity $\dot{q}^i$ (or momentum p_i). The graphs that result from plotting the trajectories in phase space are called ***phase portraits***.

Going back to our motivating example of the pendulum, we can now more clearly see the qualities/flaws of the time integrators by looking at their respective phase portraits in Fig. 1. While the pendulum's phase portrait has a characteristic structure of nested, energy-preserving orbits (since the oscillations are periodic), this was not true for the two first discrete approximations: the trajectories of explicit Euler spiraled outwards (dramatically increasing magnitude of oscillations, thus energy), while those of implicit Euler spiraled inwards. Why did some of the phase portraits look better than others? How can we preserve the closedness of the orbits without making the time integrator more complicated?

One of the key features of Lagrangian flows (*i.e.*, motions) is that they are ***symplectic***. Formally, this means that the flow preserves the canonical two-form $\Omega = dq^i \wedge dp_i$. In the two-dimensional phase plane, this directly implies that *the area of any phase-space region is preserved under the flow* (see Liouville's theorem in classical mechanics). For example, let us take a given region of initial conditions in phase-space. If we advance all these states simultaneously, the regions deforms under the flow in a way that preserves the original area as shown in Fig. 3 a cat-head shaped region: this phenomenon is called *symplecticity*. However, as seen on this same figure, explicit and implicit Euler both fail the test of symplecticity. Because orbits spiral outward under explicit Euler, a region will *expand*, and its area will increase. Conversely, implicit Euler decreases the area inside the evolving region. Preserving this property of the flow in phase space for our time integrators (that is, having them be *symplectic in a discrete sense*) is key to ensure globally correct behavior!

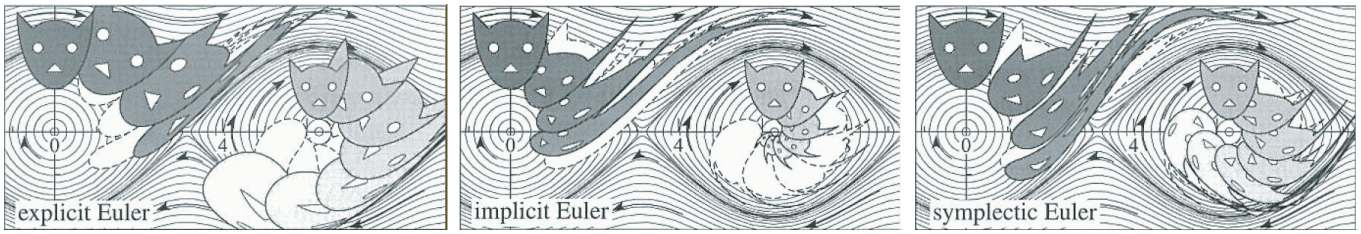

Figure 3: *Symplecticity [reproduced from [Hairer et al. 2002]]: while a continuous Lagrangian system is symplectic (that is to say, in this simple case, an area in phase space evolves along the flow without changing its area), discrete time integrators rarely share this property. From our three time integrators compared in Section 3, only the last one is symplectic. In the background, the reader will recognize the shape of the orbits obtained in Fig. 1(right).*

5 Discrete Geometric Mechanics

Having quickly reviewed classical Lagrangian mechanics in the continuous domain, we now explain how this geometric view of mechanics can elegantly be mimicked in the discrete setting.

5.1 General Idea

The driving idea behind discrete geometric mechanics is to leverage the variational nature of mechanics and to preserve this variational structure in the discrete setting. In fact, very few integrators have a variational nature: the explicit and implicit Euler methods discussed above are not variational, and not surprisingly, they both exhibited poor global behavior in the case of the pendulum. Instead of simply approximating the equations of motion to first (or higher) order as we did before, one can directly *discretize the variational principle* behind them. That is, if one designs a discrete equivalent of the Lagrangian, then discrete equations of motion can be easily derived from it by paralleling the derivations followed in continuous case. In essence, good numerical methods will come from discrete analogs to the Euler-Lagrange equations—equations that truly derive from a variational principle.

5.2 Discrete Lagrangian Dynamics

Setup The main idea is to discretize the least action principle directly rather than discretizing (5). To this end, a path $q(t)$ for $t \in [0,T]$ is replaced by a *discrete path* $q : \{t_0 = 0, t_1, \ldots, t_k, \ldots, t_N = T\}$ where $k, N \in \mathbb{N}$. Here, q_k is viewed as an approximation to $q(t_k)$.

Discrete Lagrangian The Lagrangian L is approximated on each time interval $[t_k, t_{k+1}]$ by a *discrete Lagrangian*[1] $L_d(q_k, q_{k+1}, h)$, with h being the time interval between two samples $h = t_{k+1} - t_k$ (chosen here to be constant for simplicity):

$$L_d(q_k, q_{k+1}) \approx \int_{t_k}^{t_{k+1}} L(q, \dot{q})\, dt.$$

Now, the right-hand side integral can be approximated through a one-point quadrature, i.e., by the length of the interval times the value of the integrand evaluated somewhere between q_k and q_{k+1} and with $\dot{q}$ replaced by $(q_{k+1} - q_k)/h$:

$$L_d(q_k, q_{k+1}, h) = h\, L\left((1-\alpha)q_k + \alpha q_{k+1}, \frac{q_{k+1} - q_k}{h}\right) \quad (6)$$

where $\alpha \in [0,1]$. For $\alpha = 1/2$, the quadrature is second-order accurate, while any other value leads to linear accuracy.

[1]This term could also be called an action, as it is a time integral of a Lagrangian; however, just like the term "discrete curvature" in CG refers to a small local integral of a continuous curvature, we prefer this naming convention.

Discrete Stationary Action Principle Given the discrete Lagrangian, the *discrete action functional* becomes simply a sum:

$$S_d := S_d(\{q_i\}_{i=0..N}) = \sum_{k=0}^{N-1} L_d(q_k, q_{k+1}) \approx \int_a^b L(q, \dot{q})\, dt = S(q).$$

Taking fixed-endpoint variations of this discrete action S_d, we obtain:

$$\delta S_d = \sum_{k=0}^{N-1} \Big[D_1 L_d(q_k, q_{k+1}) \cdot \delta q_k + D_2 L_d(q_k, q_{k+1}) \cdot \delta q_{k+1} \Big],$$

where $D_1 L$ (resp., $D_2 L$) denotes the partial derivative with respect to the first (resp., second) arguments of L. Reindexing the rightmost terms, and using the fixed endpoint condition $\delta q_0 = \delta q_N = 0$, one gets:

$$\delta S_d = \sum_{k=1}^{N-1} \Big[D_1 L_d(q_k, q_{k+1}) + D_2 L_d(q_{k-1}, q_k) \Big] \cdot \delta q_k.$$

Setting this variation equal to 0 and noting that each δq_k is arbitrary, we arrive at the ***discrete Euler-Lagrange (DEL) equations***

$$D_1 L_d(q_k, q_{k+1}) + D_2 L_d(q_{k-1}, q_k) = 0. \quad (7)$$

Notice that this condition only involves three consecutive positions. Therefore, for two given successive positions q_k and q_{k+1}, Eq. (7) *defines* q_{k+2}. That is, these equations of motion are actually the algorithm for an integrator! And since the DEL equations derive from the extremization of a discrete action, such an algorithm enforces the variational aspect of the motion numerically.

Link to Previous Numerical Schemes Let us go back to the pendulum case. For this system, the Lagragian (kinetic energy minus potential energy) is:

$$L(q, \dot{q}) = \frac{1}{2} L^2 \dot{q}^2 + gL\, \cos(q).$$

First, the user can convince her/himself that the Euler-Lagrange equation is indeed, Eq. (1) through a simple derivation. Second, it is also a simple (yet, interesting) exercise to verify that the symplectic Euler integrator used earlier results from the DEL equations just described, for the particular choice of $\alpha = 0$ in the quadrature rule defined in Eq. 6.

5.3 Update Rule in Phase Space

In mechanics, the initial conditions are typically specified as a position and a velocity or momentum rather than two positions, therefore it is beneficial to write (7) in a position-momentum form [West 2003]. To this end, define the momentum at time t_k to be:

$$p_k := D_2 L_d(q_{k-1}, q_k) = -D_1 L_d(q_k, q_{k+1})$$

where the second equality holds due to (7). The position-momentum form of the variational integrator discussed above is then given by:

$$p_k = -D_1 L_d(q_k, q_{k+1}) \,, \qquad p_{k+1} = D_2 L_d(q_k, q_{k+1}). \quad (8)$$

For (q_k, p_k) known, (8)(left) is an (often implicit) equation whose solution gives q_{k+1}. q_{k+1} is then substituted in (8)(right) to find p_{k+1}. This provides an update rule in phase space.

5.4 Adding Dissipation

In case of forcing and/or dissipation, the discrete action can be modified by adding the non-conservative force term and using the discrete Lagrange-d'Alembert principle [Marsden and West 2001]:

$$\delta S_d + \sum_{k=0}^{N} \left(F_d^-(q_k, q_{k+1}) \cdot \delta q_k + F_d^+(q_k, q_{k+1}) \cdot \delta q_{k+1} \right) = 0.$$

where $F_d^-(q_k, q_{k+1})$ and $F_d^+(q_k, q_{k+1})$ are discrete external forces acting respectively on the right of q_k and on the left of q_{k+1}. In other words, $F_d^-(q_k, q_{k+1}) \cdot \delta q_k + F_d^+(q_k, q_{k+1}) \cdot \delta q_{k+1}$ can be seen as a two-point quadrature of the continuous forcing term $\int_{t_k}^{t_{k+1}} F \cdot \delta q \, dt$. The *forced discrete Euler-Lagrange equations* can be expressed in a convenient, position-momentum form as follows:

$$p_k = -D_1 L_d(q_k, q_{k+1}) - F_d^-(q_k, q_{k+1}) \,,$$
$$p_{k+1} = D_2 L_d(q_k, q_{k+1}) + F_d^+(q_k, q_{k+1}).$$

This variational treatment of energy decay, despite its simplicity, has also been proven superior to the usual time integration schemes that often add numerical viscosity to get stability [West 2003].

5.5 Last Words

Variational integrators often perform better than their non-variational counterparts because they preserve the *underlying geometry* of the physical system. This has two important consequences. First, the integrators are guaranteed to be symplectic, which in practice will result in excellent energy behavior, rather than perpetual damping or blowing up. Second, they are also guaranteed to preserve discrete momenta of the system (via a discrete version of Noether's theorem). As a consequence, simulations and animations using these integrators usually have great physical and visual fidelity with low computational cost, even for dissipative systems (see [Kharevych et al. 2006] for a discussion on damping in animation). To build upon this short introduction, the reader is invited to investigate recent developments in variational integrators, such as Lie group integrators, Hamilton-Pontryagin integrators, asynchronous variational updates (where timesteps are different for each mesh element), and stochastic variational integrators.

Caveat: The reader may be misled into thinking that explicit variational schemes does not require the typical Courant-Friedrichs-Levy (CFL) condition (or equivalent) on the time step size. This is, of course, untrue: the same usual theoretical limitations of explicit schemes are still valid for symplectic explicit schemes. However, we can easily design symplectic implicit schemes that do not share this particular limitation, generally allowing for much larger time steps. Finally, we can make them of arbitrarily higher order by simply improving the quadrature rule used to convert the continuous Lagrangian into a discrete Lagrangian.

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