From Barycentric Coordinates to Whitney Forms:

Turn Your Mesh into a Computational Structure

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Big Picture

Defining Barycentric Coordinates

- super simple, you already know them
- Entending them to Edges, Faces, Tets
- called Whitney basis functions
- Deriving a whole Discrete Calculus
  - as the one you know, but on mesh

Scared?

- Don't be: it's only numbers on mesh elmts!

Barycentric Coordinates

You already know barycentric coordinates:

- given (d+1) point in \( \mathbb{R}^d \),
- way to locate point within their convex hull

\[
x = \sum_{i=0}^{d} b^i [x] v^i \text{ with } \sum_{i=0}^{d} b^i [x] = 1 \in \mathbb{R}^d
\]

Möbius invented it in 1827 – “mass points”

- coordinate-free geometry

- global vs. relative positioning
- storing numbers on vertices
- intrinsic; indpt on dim. of ambient space
- affine invariant

Computing Barycentric Coords

Question:

Given a d-simplex, find the bary. coords \( b^i [x] \)

of an inside point \( x \) for each vertex \( v^i \) such as:

\[
b^i [x] \geq 0 \quad \sum_{i=0}^{d} b^i [x] = 1 \quad \sum_{i=0}^{d} v^i b^i [x] = x
\]

Unique solution (in any dim.):

\[
b^i [x] = \frac{V_{d}[v_{0}, \ldots, v_{j-1}, x, v_{j+1}, \ldots, v_{d}]}{V_{d}[v_{0}, \ldots, v_{j-1}, v_{j}, v_{j+1}, \ldots, v_{d}]}
\]

Try it on a segment

Barycentric Basis Functions

In 1D:

\[
x = \sum_{i=0}^{d} b^i [x] v^i
\]

In 2D:

\[
u(x) = \sum_{i=0}^{d} b^i [x] u^i
\]

- linear interpolation

- Linear Finite Elements...
Extending it to Other Shapes?

What about polygons/polytopes?

Barycentric coordinates for polytopes
- ouch, not unique anymore...
- additional requirements:
  - smoothness of basis functions
  - tensor product (square → bilinear)
  - face restriction (should fit (n-1)D bc)
  - simplicity of evaluation...

Generalized B. Coords [Warren et al 04]

For convex polytope, simple expression

\[ h_j[x] = w_j[x]/\sum w_j[x] \text{, with:} \]

\[ w_j(x) = |\det (n_{j,1}, \ldots, n_{j,n})| \prod_{k=1}^{n} h_{j,k}(x) \]

Rational basis fcts of degree (F—n)
(because zero on (F—n) lines)
that extends to smooth domains!

Other Types of GBC

Other approaches:
- mean value coordinates in 2D and 3D
  - Floater, 2003/2005, positive for any star-shaped polygons
- Sibson’s – a bit complicated to compute
- blend of various coordinates
  - Hormann et al., 2003
- for any 2D polygon [Malsch et al.]
- See new paper by Ju et al. this year

None reproduces tensor prod. or generalizes to nD

So What?

What did we accomplish?
- discrete scalar fields
  - discrete sampling - values on nodes of a mesh
  - spatial interpolation to allow arbitrary evals

Where to Go from Here?
- what about vector fields?
- what about computations?
  - gradient, curl, div, laplacian, you-name-it
- keeping them mesh-intrinsic all the way?

Intrinsic Calculus on Meshes

We can bootstrap a whole discrete calculus!
- using only values on simplices
- and if needed, interpolation in space
- preview:
  - point-based scalar field
  - edge-based vector field
  - face-based vector field
  - cell-based scalar field
- deep roots in mathematics
  - algebraic topology, differential geometry
- but very simple to implement and use

Discrete Differential Quantities

HINTED in the talks before...
- they “live” at special places, as distributions
  - Gaussian curvature at vertices ONLY
  - mean curvature at edges ONLY
- they can be handled through integration
  - integration calls for k-forms (antisymmetric tensors)
  - objects that beg to be integrated (ex: \( \int f(x) \phi \))
  - k-forms are evaluated on kD set
  - 0-form is evaluated at a point,
    1-form at a curve, etc...
**Forms You Know For Sure**

**Scalar functions:** 0-forms  
**Digital Images:** 2-forms  
Incident flux on sensors (W/m²)

**Magnetic Field B:** 2-form  
Only measurement possible: \( \int B \cdot dA \)  
Any physical flux is a 2-form too

**Electrical Force E:** 1-form  
Any physical circulation is a 1-form too

*Notion of pseudo forms—see notes*

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**Exterior Calculus of Forms**

Foundation of calculus on smooth manifolds  
- Historically, purpose was to extend div/curl/grad  
  - Poincaré, Cartan, Lie, …
- Basis of differential and integral computations  
  - Highlights topological and geometrical structures
  - Modern diff. geometry, Hodge decomposition, …

A hierarchy of basic operators are defined:

- \( d, *, \wedge, b, #, i_X, L_X \)
- See [Abraham, Marsden, Ratiu], ch. 6-7

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**Discrete Exterior Calculus**

Simplicial complex only spatial structure  
- Can be 1-, 2-, or 3-manifold, flat or not

Idea: Sampling Forms on Each Simplex!  
- "Extends" the idea of point-sampling  
- Use also the "dual" of each simplex

**Discrete Differential Forms**

Discrete k-form = values on each kD set  
- Primal discrete k-form: value on each k-simplex  
- Dual discrete k-form: value on each dual (n-k)-cell  
- The rest is defined through linearity

*In math terms: chains pair with cochains  
  (natural pairing = integration)*

*In CS terms: k-form = vector of values

**Notion of Exterior Derivative**

**Stokes/Green/…** theorem:  
\[ \int_{\sigma} d\omega = \int_{\partial\sigma} \omega \]

- So \( d \) and \( \delta \) are dual  
  \( (\sigma, d\omega) = (\delta\sigma, \omega) \)

**Implementation?**

- As simple as an incidence matrix!  
- Ex: \( d(1\text{-form}) \)-incidence matrix of edges & faces  
  Bean counting:

\[ \begin{vmatrix}
|F| & |E| & |V|
\end{vmatrix} \rightarrow \begin{vmatrix}
|E| \end{vmatrix} \]

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**Exterior Derivative**

Let's try

- No "metric" needed! (no size measurement)
- Try \( d \), then \( d \) on an arbitrary form...
  - Zero, why?
  - Because \( \delta \circ d = 0 \)
  - Good: \( \text{div}(\text{curl}) = \text{curl}(\text{grad}) = 0 \) automatically
Hodge Star

Take forms to dual complex (and vice versa)
- switch values btw primal/dual
  * : \( \Omega^p \rightarrow \Omega^{n-p} \)
- “diagonal” hodge star
  \[ [\star \sigma^p] \int_{\partial \sigma^p} \alpha^p = \frac{1}{|\sigma^p|} \int_{\sigma^p} \alpha^p \]
  > again, a simple (diagonal) matrix
- now the metric enters...
- Hodge star defines accuracy
  > order of approximation of the metric

Discrete deRham Complex

Discrete calculus through linear algebra:
- simple exercise in matrix assembly
- all made out of two trivial operations:
  > summing values on simplices (\( \partial / \partial \))
  > scaling values based on local measurements (\( \bullet \))

Interpolating Discrete Forms

Whitney basis fcts to interpolate forms
- 0-forms (functions)
  > “hat” functions
- 1-forms (edge elements)
  > Whitney forms: \( \phi_{ij} = \phi_j - \phi_i \)
  > basis of 1-forms since \( \phi_{ij}(r_{ij}) = 0 \)
- 2-forms: face elmts
  > also a lot of shd. Veg
- 3-forms: constant per tet

Higher order bases for smoother interp.

Why DEC is New

Not quite like:
- Finite Elements
  > use nodes and cells only
  > tries to enforce local relationships globally well
- Finite Differences
  > sorta local polynomial fitting, loses invariants
- Finite Volumes
  > use cells and nodes only
  > good at local relations, often had at global ones
Why DEC is Limited

It Does Not Substitute For:
- Numerical Analysis
  - accuracy and convergence rate still need careful study
- Good Meshing Tools
  - bad mesh? bad results, guaranteed...
  - stay tuned; we’ll address the issue in the last talk
- Good hacking skills
  - see Building your own DEC at home in notes

Why DEC is Good

If You Ask Me:
- basic discrete operators, consistently derived
  - easy to compute given a discrete mesh
  - separating topology from geometry
    - helps narrowing down where accuracy is lost
  - conservation laws can be preserved exactly
    - preserving structures at the discrete level!
    - applicable to a variety of problems
  - good foundations for further studies
    - can be used as basis for ‘simplex sampling’

Other Related Research Areas

Variational Integrators
- principle of least-action is crucial
  - motion is a geodesic if we use action as "metric"
  - preserve invariants and symmetries
- "not accurate?", urban legend, simply untrue

Discrete Differential Euclidean Geometry
- it all ties up through the use of connections
- but it will be for another day...

Take-Home Message

Geometric Approach to Computations
- discrete setup acknowledged from the get-go
- choice of proper habitat for quantities
- whole calculus built using only:
  - boundary of mesh elements
  - scaling by local measurements
- preserving structural identities
  - they are not just abstract concepts:
    - they represent defining symmetries